

# Star graphs: threaded distance trees and E-sets

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## Abstract

The distribution of distances in the star graph  $ST_n$ , ( $1 < n \in \mathbf{Z}$ ), is established, and subsequently a threaded binary tree is obtained that realizes an orientation of  $ST_n$  whose levels are given by the distances to the identity permutation, via a pruning algorithm followed by a threading algorithm. In the process, the distributions of distances of the efficient dominating sets of  $ST_n$  are determined.

## 1 Introduction

The *star graph*  $ST_n$ , ( $1 < n \in \mathbf{Z}$ ), is the Cayley graph of the symmetric group  $S_n$  with set of generators  $\Theta_n = \{(1\ i),\ i = 2, \dots, n\}, ([1, 2])$ . The *weight* of a vertex  $u$  of  $ST_n$  is its distance to the identity-permutation vertex  $12 \dots n$ . In this work, based on DIMACS Technical Report 2001-05, the weight distributions of certain subsets  $C$  of  $ST_n$  are determined, including that of  $ST_n$  itself. Theorems 8 and 6 below attain these objectives. (A variation of Theorem 6 was obtained in a different fashion in [6]).

An independent set  $C$  of vertices in a graph is an efficient dominating set [4], or E-set [3], or 1-perfect codes [5], if each vertex not in  $C$  is adjacent to exactly one vertex of  $C$ . In Section 5, we determine the weight distributions of these E-sets; see Theorem 8 and subsequent remark. In obtaining this, we use a binary directed tree  $\Lambda_n = \Lambda(ST_n)$  whose arcs are of two types: **(1)** horizontal, left-to-right, arcs; **(2)** vertical, top-to-bottom, arcs, (as in the subsequent figures). In Section 6, we extend  $\Lambda_n$  to an orientation  $\Gamma_n$  of  $ST_n$ , (that is: an oriented graph  $\Gamma_n$ ). Moreover, the graphs  $\Gamma_n$  form a nested sequence that converges to a universal graph  $\Gamma_\infty$  associated to the infinite star graph  $ST_\infty$ .

## 2 Definition and examples of $\Lambda_n$

Let  $n > 1$  and let  $\Sigma \in S_n$ . We write  $\Sigma = \sigma_1 \sigma_2 \dots \sigma_n$ , where  $\Sigma(i) = \sigma_i$ , for  $i = 1, 2, \dots, n$ . A cycle  $(\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_r})$  of the permutation  $\Sigma$  is given by  $\Sigma(\sigma_{i_j}) = \sigma_{i_{j+1}}$ , for  $j = 1 \dots r$ , where  $j + 1$  is taken as 1 if  $j = r$ . Then,  $\Sigma$  has *length*  $r$ . Now,  $\Sigma$  is said to be *proper* if  $r > 1$ . The *cycle structure*  $\Pi(\Sigma)$  of  $\Sigma = \sigma_1 \sigma_2 \dots \sigma_n$  is defined as the set of proper cycles of  $\Sigma$ .

Two vertices  $\Sigma^1$  and  $\Sigma^2$  of  $ST_n$ , with 1 in cycles  $\tau^1$  of  $\Sigma^1$  and  $\tau^2$  of  $\Sigma^2$  of the same length, have a common 1-invariant cycle structure if there is  $\Phi \in S_n$  with  $\Phi(\Sigma^1) = \Sigma^2$  inducing a 1-1 correspondence  $\Phi^* : \Pi(\Sigma^1) \rightarrow \Pi(\Sigma^2)$  sending  $\tau^1$  onto  $\tau^2$  and with each  $\tau \in \Pi(\Sigma^1)$  and  $\Phi(\tau) \in \Pi(\Sigma^2)$  having the same length. We say that  $\Sigma^2$  has the 1-invariant cycle structure, (or 1-ics), of  $\Sigma^1$ . Each vertex  $u$  of  $\Lambda_n$  is written

$$\begin{array}{c} w(u), c(u) \\ \hline \Sigma(u) \end{array}$$

where **(a)**  $\Sigma(u) = \sigma_1 \dots \sigma_{i-1}$  is shorthand for a permutation  $\sigma_1 \sigma_2 \dots \sigma_n$  of  $12 \dots n$  having  $i$  as the smallest index in  $\{2, \dots, n\}$  satisfying  $\sigma_j = j$ , for  $i \leq j \leq n$ , and  $\sigma_j \neq j$  for  $1 < j < i$ ; **(b)**  $w(u)$  is the weight of  $\Sigma(u)$ ; **(c)**  $c(u)$  is the cardinality of the set  $S(u)$  of permutations having the 1-ics  $\Pi(u)$  of  $\Sigma(u)$ .

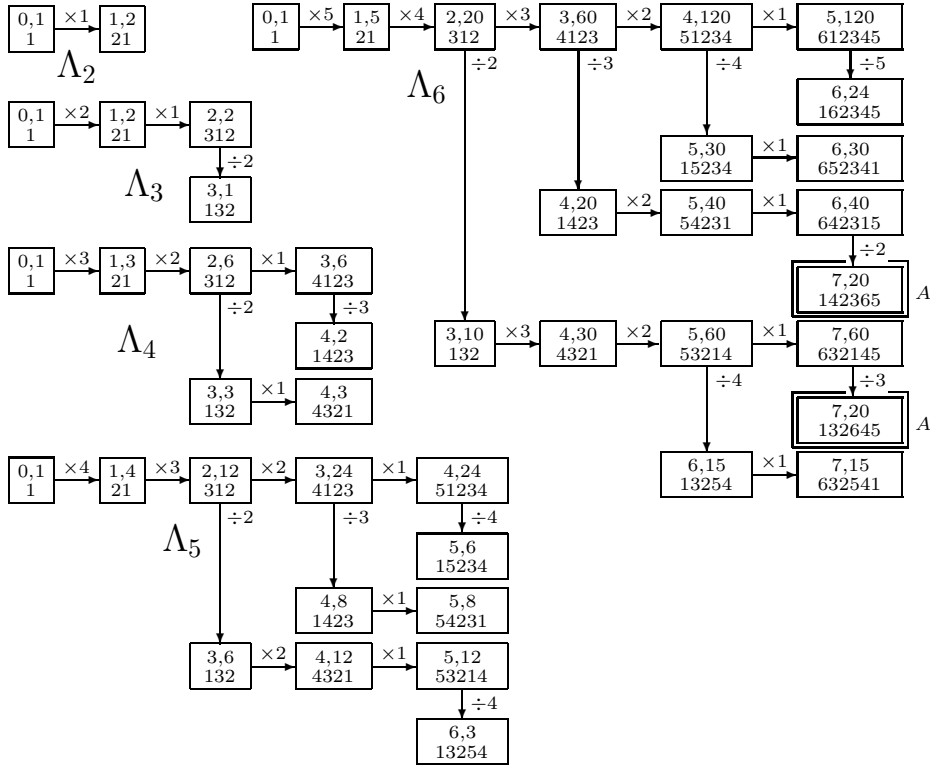


Figure 1: Representations of  $\Lambda_n$ , for  $n = 2, 3, 4, 5, 6$ .

The *string length*  $\lambda(\Sigma(u))$  of  $u$  is defined as the number of entries ( $\leq n$ ) of  $\Sigma(u)$ . Given an arc  $e$  of  $\Lambda_n$ , let  $u_e$  and  $u^e$  be the tail and the head of  $e$ , respectively. The two types of arcs in  $\Lambda_n$  are selected as follows:

(1) arcs  $e$  with  $\lambda(\Sigma(u^e)) = 1 + \lambda(\Sigma(u_e))$ , as shown in Figures 1–3, indicated with a multiplicative operator  $\times m_e$ , where  $c(u^e) = c(u_e) \times m_e$ , noticing that  $\sigma_1(u^e) \neq 1$ ;

(2) the remaining arcs  $f$ , indicated with a divisive operator  $\div d_f$  determined by  $c(u^f) = c(u_f) \div d_f$ , noticing that  $\sigma_1(u^f) = 1$  and that there is not an arc  $e$  of type (1) with  $u_e = u^f$  and  $u^e = u_f$ .

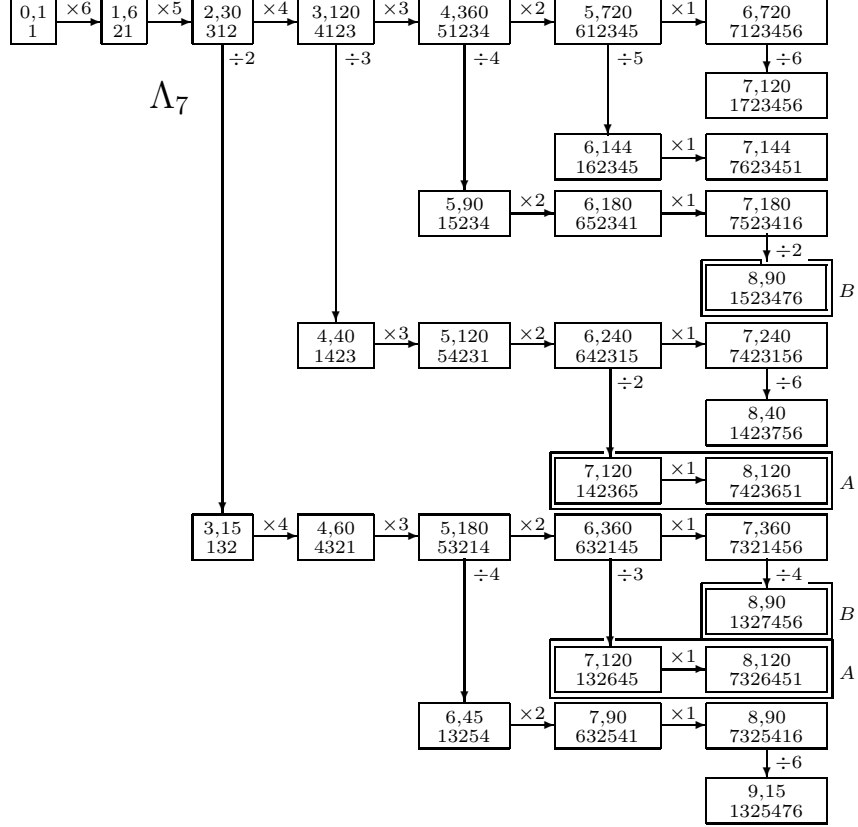


Figure 2: Representation of  $\Lambda_7$ .

An additional requirement in the definition of  $\Lambda_n$  is that it is a rooted tree; its root is denoted  $u_0 = u_0^n$ , with  $w(u_0) = 0$ ,  $c(u_0) = 1$  and  $\Sigma(u_0) = 1$ .

Given a maximal horizontal directed path, (or mhd),  $P$  of  $\Lambda_n$ , the *depth* of  $P$  is the number of vertical arcs of  $\Lambda_n$  preceding  $P$  from  $u_0$ .

**Examples.** Figures 1–3 contain the representations of  $\Lambda_n$  for  $n = 2, \dots, 8$  (with the root of  $\Lambda_8$  in Figure 3 squeezed on the bottom left), where pairs of encased mhd's  $U_I, V_I$ , either improper, (i.e. consisting of one vertex), or proper, and indicated with a common capital letter  $I = A, B, \dots$  on their right, have corresponding vertex sets  $\{u_j^I\}, \{v_j^I\}$  representing each a complete set of permutations with a common 1-ics, and thus having a common cardinality.

In fact, to determine the weight distribution of  $ST_n$ , the Pruning Algorithm of section 3 below will leave only one of these encased mhd's with a common capital letter  $I$ , provided a denomination  $u_{i_0 i_1 \dots i_{(j-2)} i_{(j-1)}}$  for each vertex of  $\Lambda_{n+1}$

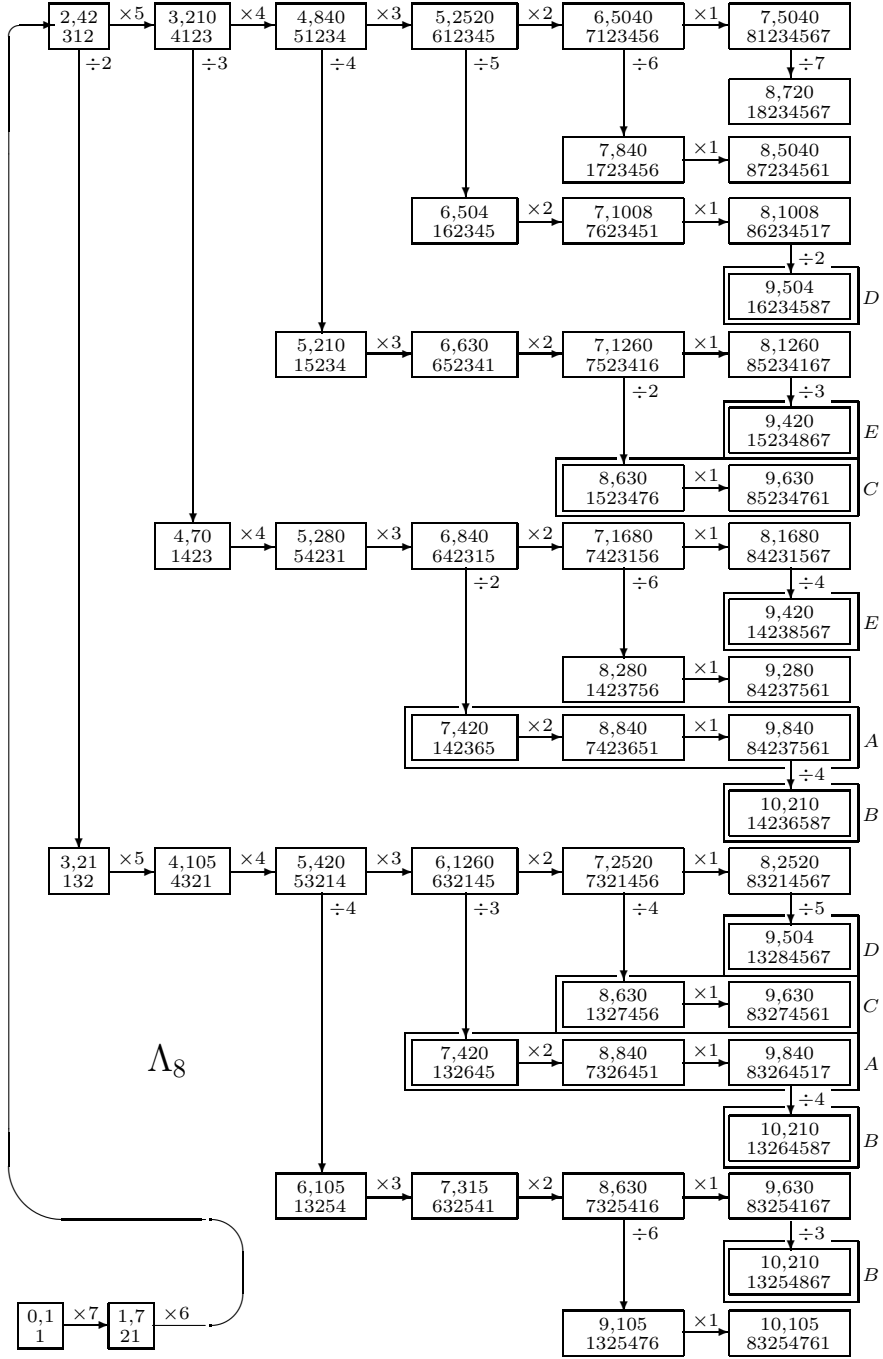


Figure 3: Representation of  $\Lambda_8$ .

is given via the following inductive definition of Axiom  $(j)$ , for  $j = 0, 1 \dots \lfloor n/2 \rfloor$ , and exemplified in Figure 4, showing the strings  $i_0 i_1 \dots i_{(j-2)} i_{(j-1)}$  in those denominations, for the vertices of  $\Lambda_8$  in their positions in Figure 3. **Axiom (0):** there is an mhd  $u_0 u_1 \dots u_n$  of depth 0 in  $\Lambda_{n+1}$ . **Axiom  $(j)$ :** for each  $u_{i_0 i_1 \dots i_{(j-2)} i_{(j-1)}}$  as in property  $(j-1)$  with  $i_{j-2} + 1 < i_{j-1}$ , there is a vertical arc  $u_{i_0 i_1 \dots i_{(j-1)}} u_{i_0 i_1 \dots i_{(j-1)} i_{(j-1)}}$  and an mhd  $p$  from  $u_{i_0 i_1 \dots i_{(j-1)} i_{(j-1)}}$  to  $u_{i_0 i_1 \dots i_{j-1} n}$  whose depth is  $j$  in  $\Lambda_{n+1}$ .

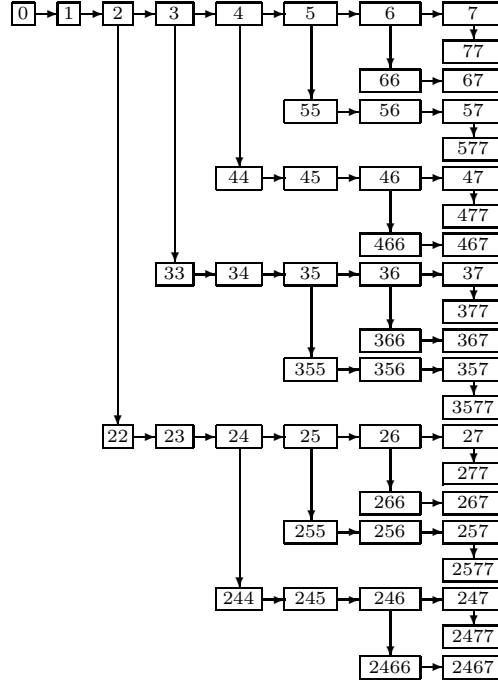


Figure 4: Index string representation of  $\Lambda_8$ .

### 3 Redefinition and pruning of $\Lambda_n$

If a vertex  $u$  of  $\Lambda_n$  is the tail of a horizontal (vertical) arc  $e$  indicated  $\times m_e$  ( $\div d_e$ ) then we write  $m_u = m_e$  ( $d_u = d_e$ ). If  $u$  is not the tail of an arc, then we write  $m_u = 0$ , ( $d_u = 0$ ). The following redefinition allows to consider  $\Lambda_n$  as a subdigraph of  $\Lambda_{n+1}$  for any  $n$ , so the limit  $\Lambda_\infty$  of the nested sequence  $\{\Lambda_n; n > 0\}$  of binary directed trees makes sense: replace the indication  $\times m_u = \times m_e$  of the horizontal arc  $e$  of  $\Lambda_n$  departing from a tail  $u$  of an arc in  $\Lambda_n$  by  $\bullet \ell_u$ , where  $\ell_u = n - m_u$  and  $\bullet$  is the operation given by  $c(u) \bullet \ell_u = c(u) \times (n - m_u)$ ; let  $\Lambda_n^\bullet$  be the resulting indicated digraph; redefine  $\Lambda_n = \Lambda_n^\bullet$ . The new indications for the horizontal arcs allow now the containment of indicated subdigraphs. Incidentally, the cardinalities of the set of vertices of the resulting  $\Lambda_\infty$  that are

tails of arcs indicated  $\bullet_i$  form a Fibonacci sequence according to the increasing values of  $i = 0, 1, \dots$ . This is apparent from the number of vertices in the successive columns from left to right, in the representations of the  $\Lambda_n$ 's as in Figures 1–3, for increasing values of  $n = 2, 3, \dots$ .

**Pruning Algorithm.** The vertices  $u_{i_0 i_1 \dots i_j}$  of  $\Lambda_n$  are treated first in the increasing order of their string lengths  $j+1$  and then, for each fixed string length  $j+1$ , in the lexicographical order of their subindex strings  $i_0 i_1 \dots i_j$ , namely:

$$u_0, u_1, \dots, u_{n-1}, u_{2,2}, u_{2,3}, \dots, u_{2,n-1}, u_{3,3}, \dots, \\ u_{n-1,n-1}, u_{2,4,4}, \dots, u_{2,4,6,6}, \dots$$

Each such a vertex  $u = u_{i_0 i_1 \dots i_j}$  has the following fields associated with it: **(1)** the notation  $u = u_i^w = u_{i_0 i_1 \dots i_j}^{w(u)}$ , where  $w(u) = w(\Sigma(u))$ ; **(2)** the notation  $\Sigma(u)$  of the corresponding permutation of  $\{1, \dots, n\}$  associated to  $u$ ; **(3)** the 1-ics  $\Pi(u)$  of  $\Sigma(u)$ ; **(4)** the number  $\ell_u = n - m_u$ ; **(5–7)** either a blank in each of the three cases (5), (6) and (7), if  $u$  is the first or second vertex of an mhdps, or: **(5)** the notation  $\Sigma[u]$  of the permutation obtained from  $\Sigma(u)$  by permuting  $\sigma_1$  and  $\sigma_k = 1$ , ( $k \neq 1$ ); **(6)** the 1-ics  $\Pi[u]$  of  $\Sigma[u]$ ; **(7)** a tuple  $C(u) = s_1, \dots, s_h$  composed by the orders  $s_j$  of the cycles composing  $\Pi[u]$ ; **(8)** the number  $d_u$  expressed as a product  $b_u a_u$ , where **(8a)**  $a_u = i_j - i_{j-1}$  under the convention  $i_{-1} = 0$  and **(8b)**  $b_u \neq 0$  if the value of item (7) above is not a blank and the resulting tuple  $C(u)$  was not present in previously treated vertices of  $\Lambda_n$ ;  $b_u = 0$ , otherwise.

The Pruning Algorithm consists in determining these fields for the vertices of  $\Lambda_n$ , in the prescribed order. This allows a partial reconstruction of  $\Lambda_n$  in the form of the maximal subdigraph  $\Lambda'_n$ , which accepts and copies all the vertices and arcs of  $\Lambda_n$  into  $\Lambda'_n$  except for one case: If  $b_u = 0$  and there is a vertical arc  $e$  whose tail is  $u$ , then  $e$  is not copied from  $\Lambda_n$  into  $\Lambda'_n$ ; this is interpreted as the pruning of  $e$  and descendant vertices and arcs (performed to avoid repetitions of mhdps's, as in the encased mpdh's having a common capital-letter indication at their right in Figures 1–3).  $\square$

Let  $\rho_n$  be the relation defined on the vertex set of  $ST_n$  by  $u\rho_n v$  if and only if  $u$  and  $v$  represent permutations with a common 1-ics. The following second redefinition of  $\Lambda_n$  allows to have its vertex set in bijective correspondence with the family of equivalence classes of  $ST_n$  under  $\rho_n$ , which in turn allows to use  $\Lambda_n$  in computing the weight distribution of  $ST_n$ : perform the Pruning Algorithm of  $\Lambda_n$ , whose output is a maximal subdigraph  $\Lambda'_n$  of  $\Lambda_n$  in which there are not pairs of mhdps's  $v_0 v_1 \dots v_s$  and  $v'_0 v'_1 \dots v'_s$  of the same string length  $s$  with corresponding vertices  $v_i$  and  $v'_i$  having common 1-ics  $\Pi(v_i) = \Pi(v'_i)$ , for  $i = 0, 1, \dots, s$ ; redefine  $\Lambda_n = \Lambda'_n$ . We still have  $\Lambda_n$  as a subdigraph of  $\Lambda_{n+1}$  for every  $n$ , so a  $\Lambda_\infty$  persists.

**Example.** The algorithm yields the list  $\mathcal{P}_9$  for  $n = 9$ , (commas are deleted in subindices  $i$  of  $u_i^w = u(i, w)$  in item 1 and in the tuples  $C(u)$  in item 7; the  $\Pi(u)$  and  $\Pi[u]$  are shown to the right of their corresponding  $\Sigma(u)$  and  $\Sigma[u]$ ):

$u(i, w)$	$\Sigma(u)\Pi(u)$	$\ell_u$	$\Sigma[u]\Pi[u]$	$C(u)$	$b_u a_u$
$u(0, 0)$	1				00
$u(1, 1)$	21(12)	2		1	01
$u(2, 2)$	312(132)	3	132(32)	2	12
$u(3, 3)$	4123(1432)	4	1423(432)	3	13
$u(4, 4)$	51234(15432)	5	15234(5432)	4	14
$u(5, 5)$	612345(165432)	6	162345(65432)	5	15
$u(6, 6)$	7123456(1765432)	7	1723456(765432)	6	16
$u(7, 7)$	81234567(18765432)	8	18234567(8765432)	7	17
$u(8, 8)$	912345678(198765432)	9	192345678(98765432)	8	18
$u(22, 3)$	132(23)	3			00
$u(23, 4)$	4321(14.32)	4			01
$u(24, 5)$	53214(154.32)	5	13254(54.32)	22	22
$u(25, 6)$	632145(1654.32)	6	132645(654.32)	23	13
$u(26, 7)$	7321456(17654.32)	7	1327456(7654.32)	24	14
$u(27, 8)$	83214567(187654.32)	8	13284567(87654.32)	25	15
$u(28, 9)$	932145678(1987654.32)	9	132945678(987654.32)	26	16
$u(33, 4)$	1423(432)	4			00
$u(34, 5)$	54231(15.432)	5			01
$u(35, 6)$	642315(165.432)	6	142365(65.432)	32	02
$u(36, 7)$	7423156(1765.432)	7	1423756(765.432)	33	23
$u(37, 8)$	84231567(18765.432)	8	14238567(8765.432)	34	14
$u(38, 9)$	942315678(198765.432)	9	142395678(98765.432)	35	15
$u(44, 5)$	15234(5432)	5			00
$u(45, 6)$	652341(16.5432)	6			01
$u(46, 7)$	7523416(176.5432)	7	1523476(76.5432)	42	02
$u(47, 8)$	85234167(1876.5432)	8	15234867(876.5432)	43	03
$u(48, 9)$	952341678(19876.5432)	9	152349678(9876.5432)	44	24
$u(55, 6)$	162345(65432)	6			00
$u(56, 7)$	7623451(17.65432)	7			01
$u(57, 8)$	86234517(187.65432)	8	16234587(87.65432)	52	02
$u(58, 9)$	962345178(1987.65432)	9	162345978(987.65432)	53	03
$u(66, 7)$	1723456(765432)	7			00
$u(67, 8)$	87234561(18.765432)	8			01
$u(68, 9)$	972345618(198.765432)	9	172345698(98.765432)	62	02
$u(77, 8)$	18234567(8765432)	8			00
$u(78, 9)$	982345671(19.8765432)	9			01
$u(88, 9)$	192345678(98765432)	9			00
$u(244, 6)$	13254(54.32)	5			00
$u(245, 7)$	632541(16.54.32)	6			01
$u(246, 8)$	7325416(176.54.32)	7	1325476(23.45.67)	222	32
$u(247, 9)$	83254167(1876.54.32)	8	13254867(23.45.687)	223	13
$u(248, 10)$	932541678(19876.54.32)	9	132549678(23.45.6987)	224	14
$u(255, 7)$	132645(654.32)	6			00
$u(256, 8)$	7326451(17.654.32)	7			01
$u(257, 9)$	83264517(187.654.32)	8	13264587(23.465.78)	232	02
$u(258, 10)$	932645178(1987.654.32)	9	132645978(23.465.798)	233	13
$u(266, 8)$	1327456(7654.32)	7			00
$u(267, 9)$	83274561(18.7654.32)	8			01
$u(268, 10)$	932745618(198.7654.32)	9	132745698(23.4765.89)	242	02
$u(277, 9)$	13284567(87654.32)	8			00
$u(278, 10)$	932845671(19.87654.32)	9			01
$u(288, 10)$	132945678(987654.32)	9			00
$u(366, 8)$	1423756(765.432)	7			00
$u(367, 9)$	84237561(18.765.432)	8			01
$u(368, 10)$	942375618(198.765.432)	9	142375698(243.576.89)	332	02
$u(377, 9)$	14238567(8765.432)	8			00
$u(378, 10)$	942385671(19.8765.432)	9			01
$u(388, 10)$	142395678(98765.432)	9			00
$u(488, 10)$	152349678(9876.5432)	9			00
$u(2466, 9)$	1325476(76.54.32)	7			00
$u(2467, 10)$	83254761(18.76.54.32)	8			01
$u(2468, 11)$	932547618(198.76.54.32)	9	132547698(23.45.67.89)	2222	42
$u(2477, 10)$	13254867(876.54.32)	8			00
$u(2478, 11)$	932548671(19.876.54.32)	9			01
$u(2488, 11)$	132549678(9876.54.32)	9			00
$u(2588, 11)$	132645978(987.654.32)	9			00
$u(24688, 12)$	$u(132547698(98.76.54.32))$	9			00

This list  $\mathcal{P}_n$  generalizes to the patterns expressed in the following theorem. For  $u = u_{i_0 i_1 \dots i_j}$  in  $\Lambda_n$ , let  $\ell_u = \ell_{i_0 i_1 \dots i_j}$ , etc.

**Theorem 1** *Let  $i_{-1} = 0$  and let  $t_k = i_k - i_{k-1}$ , for  $k = 0, 1, \dots, j-1$ . Then: (1) the 1-ics  $C(u)$  in the penultimate field of the line associated to a vertex  $u = u_{i_0 i_1 \dots i_j}$  in  $\mathcal{P}_n$  is of the form  $t_0, t_1, \dots, t_j$ , where the order of the integers  $t_k$  is irrelevant; (2) the vertices  $u_{i_0 i_1 \dots i_j}$  of  $\Lambda_n$ , (remaining after applying the Pruning Algorithm), have subindex strings  $i_0 i_1 \dots i_j$  completely determined by the following conditions:*

- (a)  $0 \leq i_0 \leq n-1$ ; (b) if  $j > 0$ , then  $2 \leq i_0$ ;
- (c)  $t_k \leq t_{k+1}$ , for  $k = 0, \dots, j-2$ ; (d)  $i_{j-1} \leq i_j$ ;

- (3) the weight  $w(u)$  of a vertex  $u = u_{i_0 i_1 \dots i_j}$  of  $\Lambda_n$  is  $w(u) = w(u_{i_0 i_1 \dots i_j}) = i_j + j$ ;
- (4) the number  $\ell_u$  associated to a vertex  $u_{i_0 i_1 \dots i_j}$  of  $\Lambda_n$  is  $\ell_u = \ell_{i_0 i_1 \dots i_j} = i_j + 1$ ;
- thus, the corresponding multiplicative factor  $m_u$  is  $m_u = m_{i_0 i_1 \dots i_j} = n - i_j - 1$ ;
- (5) the divisive-operator number  $d_u = b_u \cdot a_u$  has  $a_u = t_j$ ; moreover,  $b_u > 0$  if and only if either  $j = 0$  and  $i_0 > 1$  or  $j > 0$  and  $2 \leq i_0 \leq t_1 \leq t_2 \leq \dots \leq t_j$ ; furthermore, if  $b_u > 0$ , then  $b_u = 1$ , unless  $i_0 = t_1 = t_2 = \dots = t_j$ , in which case  $b_u = j + 1$ .

## 4 The weight distribution of $ST_n$

To compute the weight distribution of  $ST_n$ , a table  $\mathcal{T}_n$  constructed from the resulting pruned version of  $\Lambda_n$  and satisfying the following additional conditions will be used: **(a)** the subindex strings  $i_0 i_1 \dots i_j$  of the vertices  $u_{i_0 i_1 \dots i_j}$  of  $\Lambda_n$  are distributed on columns according to their weights; **(b)** each row is to contain the subindex strings of the vertices of an mhdP  $P$  of  $\Lambda_n$ , given from left to right according to the orientation of  $P$ ; **(c)** each mhdP is presented in lexicographical order in its containing row; **(d)** the rows of each complete set of common-depth mhdP's are presented contiguously and in the decreasing order of their path lengths, thus forming upper triangular matrices, because of item (a), above; **(e)** these upper triangular matrices are given from top to bottom in the increasing order of their depths.

**Example.**  $\mathcal{T}_{11}$  is as follows, where  $a = 10$  and  $b = 11$ , vertices with  $j = 2$  and  $i_0 = 3$  previous to 366 do not appear since they were pruned, and one additional row should be added for the 15-th column, containing solely the string 2468aa, (which, for insufficient margin, remained excluded):

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	2	3	4	5	6	7	8	9	a	2a			
			22	23	24	25	26	27	28	29	3a			
				33	34	35	36	37	38	39	4a			
					44	45	46	47	48	49	5a			
						55	56	57	58	59	6a			
							66	67	68	69	7a			
								77	78	79	8a			
									88	89	9a			
										99	aa			



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
						244	245	246	247	248	249	24a		
							255	256	257	258	259	25a		
								266	267	268	269	26a		
									277	278	279	27a		
										288	289	28a		
											299	29a		
												2aa		
								366	367	368	369	36a		
									377	378	379	37a		
										388	389	38a		
											399	39a		
										488	489	48a		
											499	49a		
												4aa		
												5aa		
									2466	2467	2468	2469	246a	
										2477	2478	2479	247a	
											2488	2489	248a	
												2499	249a	
													24aa	
											2588	2589	258a	
												2599	259a	
													25aa	
													26aa	
													369a	
													36aa	
												24688	24689	2468a
													24699	2469a
														246aa
														247aa

Each vertex  $u = u_{i_0 i_1 \dots i_j} = i_0 i_1 \dots i_j$  of  $\Lambda_n$ , reachable from  $u_0 = 0$  by a path  $P$ , has associated cardinality  $c(u) = M/(A.B)$ , where: **(a)**  $M$ , (respectively  $A$ ), is the product of the numbers  $m_{i_0 i_1 \dots i_j} = n - i_j - 1$ , (resp.  $a_{i_0 i_1 \dots i_j} = t_j = i_i - i_{j-1}$ ), of all tails  $i_0 i_1 \dots i_j$  of horizontal, (resp. vertical), arcs in  $P$ ; **(b)**  $B$  is the product of all the numbers  $b_{i_0 i_1 \dots i_j}$  of tails  $i_0 i_1 \dots i_j$  of vertical arcs in  $P$  with  $i_0 = t_1 = \dots = t_j$ .

A procedure to compute the path from  $u_0$  to any given vertex  $u$  of  $\Lambda_n$ , performed by going backwards from  $i_0 i_1 \dots i_j$  to 0 by means of table  $\mathcal{T}_n$ , consists of the following steps: **(1)** set  $u = i_0 i_1 \dots i_j$ ; **(2)** if  $u$  is not the first vertex of an mhd, then go backwards through the vertices of the mhd containing  $i_0 i_1 \dots i_j$ ; **(3)** once arrived to the first vertex  $v$  of an mhd, or in the case that  $u = v$  is such a first vertex, consider its vertical predecessor, that is the tail  $z$  of the vertical arc in  $\Lambda_n$  with head  $v$ , (which is in the column previous to that containing  $v$ ); **(4)** set  $u = z$  and repeat item (2); **(5)** continue until vertex 0 is reached.

**Example.** Let  $i_0 i_1 \dots i_j = 2468aa$  be the vertex of  $\Lambda_{11}$  whose weight is 15, (the one left out of the encased table above). This is the first (and only) vertex of its (improper) mhd. Its vertical predecessor, in column 14, is 2468a. This is preceded horizontally by 24689 and this by 24688, in respective columns 13 and 12. The vertical predecessor of 24688 is 2468, in column 11, preceded horizontally by 2467 and this by 2466, in respective columns 10 and 9. The vertical predecessor of 2466 is 246, in column 8, preceded horizontally by 245 and this by 244, in respective columns 7 and 6. The vertical predecessor of 244 is 24, in column 5, preceded horizontally by 23 and this by 22, in respective columns 4 and 3. The vertical predecessor of 22 is 2, in column 2, preceded horizontally by 1 and this by 0 =  $u_0$ , in respective columns 1 and 0. Thus we get the following path, with commas replaced by superindices  $m_u$ , for horizontal-arc tails  $u$ , and subindices  $d_u$ , for vertical-arc tails  $u$ , respectively:

$$0^{10}1^92_222^823^724_4244^6245^5246_62466^42467^32468_824688^224689^12468a_{10}2468aa.$$

We arrive at  $c(2468aa) = 9 \times 7 \times 5 \times 3$ .

This generalizes to the following statement.

**Theorem 2** *If  $n = 2k + 1$ , then the paths realizing the diameter  $D(ST_n)$  of  $ST_n$  and starting at  $12 \dots n$  end up at exactly  $(n - 2)(n - 4) \dots 3$  vertices  $u$  of the form  $\Sigma(u) = \sigma_1 \sigma_2 \dots \sigma_n$ , with  $\sigma_1 = 1$  and  $\Pi(u)$  expressible as a product of  $k + 1$  independent transpositions.*

A string  $i_0 i_1 \dots i_j$  is said to be admissible if  $u_{i_0 i_1 \dots i_j}$  is a vertex of  $\Lambda_\infty$ . Given a positive integer  $\omega \leq D(ST_n)$ , we want first to find an expression for the cardinality of the set  $V_\omega$  of vertices of  $\Lambda_\infty$  having  $\omega$  as their weight in  $ST_{\omega+1}$ . Toward this end, we start exemplifying some sequences of admissible strings for lower values of  $\omega$ , where subindex strings  $i_0 i_1 \dots u_j$  of vertices  $u_{i_0 i_1 \dots u_j}$  are expressed in a suitable order without commas and employing the following shorthand dot-notation rule for certain subsequences: let  $i_0 i_1 \dots i_{k-1} i_k \cdot i_{k+1} \dots i_{j-1} i_j$  stand for the subsequence composed by all the admissible strings  $i_0 i_1 \dots i_{k-1} i_\ell \cdot i_{k+1} \dots i_{j-1} i_j$  in  $\Lambda_n$  with  $i_\ell \geq i_k$ , for  $k \leq \ell < j$ .

**Examples.** Some subsequences of admissible strings in  $\Lambda_\infty$  are:

2.2	=	{22}	2.i <sub>1</sub>	=	{2i <sub>1</sub> , 3i <sub>1</sub> , ..., i <sub>1</sub> i <sub>1</sub> }	i <sub>1</sub> > 2
24.4	=	{244}	24.i <sub>2</sub>	=	{24i <sub>2</sub> , 25i <sub>2</sub> , ..., 2i <sub>2</sub> i <sub>2</sub> }	i <sub>2</sub> > 4
36.6	=	{366}	36.i <sub>2</sub>	=	{36i <sub>2</sub> , 37i <sub>2</sub> , ..., 3i <sub>2</sub> i <sub>2</sub> }	i <sub>2</sub> > 6
246.6	=	{2466}	246.i <sub>3</sub>	=	{246i <sub>3</sub> , 247i <sub>3</sub> , ..., 24i <sub>3</sub> i <sub>3</sub> }	i <sub>3</sub> > 6
369.9	=	{3699}	369.i <sub>3</sub>	=	{369i <sub>3</sub> , 36ai <sub>3</sub> , ..., 36i <sub>3</sub> i <sub>3</sub> }	i <sub>3</sub> > 9

For  $\omega = 0, 1, \dots, 15 = f$  we can express  $V_\omega$  as follows, where hexadecimal notation is used:

$V_0 = \{0\}$				$V_3 = \{3,$	2.2}
$V_1 = \{1\}$				$V_4 = \{4,$	2.3}
$V_2 = \{2\}$				$V_5 = \{5,$	2.4}
$V_6 = \{6,$	2.5,	24.4}			
$V_7 = \{7,$	2.6,	24.5}			
$V_8 = \{8,$	2.7,	24.6,	36.6}		
$V_9 = \{9,$	2.8,	24.7,	36.7,		
		246.6}			
$V_a = \{a,$	2.9,	24.8,	36.8,	48.8,	
		246.7,	257.7}		
$V_b = \{b,$	2.a,	24.9,	36.9,	48.9,	
		246.8,	257.8,	268.8}	
$V_c = \{c,$	2.b,	24.a,	36.a,	48.a,	5a.a,
		246.9,	257.9,	268.9,	279.9,
		369.9			
		2468.8}			
$V_d = \{d,$	2.c,	24.b,	36.b,	48.b,	5a.b,
		246.a,	257.a,	268.a,	279.a,
		369.a,	37a.a,		28a.a,
		2468.9,	2579.9}		
$V_e = \{e,$	2.d,	24.c,	36.c,	48.c,	5a.c,
		246.b,	257.b,	268.b,	6c.c,
		369.b,	37a.b,	38b.b,	279.b,
		2468.a,	2579.a,	268a.a}	28a.b, 29b.b,
$V_f = \{f,$	2.e,	24.d,	36.d,	48.d,	5a.d,
		246.c,	257.c,	268.c,	6c.d,
		369.c,	37a.c,	38b.c,	279.c,
		2468.b,	2579.b,	268a.b,	28a.c, 29b.c, 2ac.c,
		2468a.a}			

The ten last  $V_\omega$  here are expressible as:

$V_6=\{6, 2.5, 2.44\}$	$V_9=\{9, 2.8, 2.47, 2.466\}$
$V_7=\{7, 2.6, 2.45\}$	$V_a=\{a, 2.9, 2.48, 2.467\}$
$V_8=\{8, 2.7, 2.46\}$	$V_b=\{b, 2.a, 2.49, 2.468\}$
$V_c=\{c, 2.b, 2.4a, 2.469, 2.4688\}$	
$V_d=\{d, 2.c, 2.4b, 2.46a, 2.4689\}$	
$V_e=\{e, 2.d, 2.4c, 2.46b, 2.468a\}$	
$V_f=\{f, 2.e, 2.4d, 2.46c, 2.468b, 2.468aa\}$	

Let  $V_\omega^{i_0}$  be the subset of strings of  $V_\omega$  starting at  $i_0$ . We draw the following conclusions, where the dot-notation rule is used: **(1)**  $\lambda = 1$  happens in  $V_\omega$  just for each subsequence  $\omega \geq 0$ , and only in  $V_\omega^1$ ; **(2)**  $\lambda = 2$  happens in  $V_\omega$  for the members of  $2.(\omega - 1)$ , where  $\omega \geq 3$ , and only in  $V_\omega^2$ ; **(3)**  $\lambda = 3$  happens in  $V_\omega$ : **(a)** for the members of  $24.(\omega - 2)$ , where  $\omega \geq 6$ , and only in  $V_\omega^2$ ; **(b)** for the members of  $36.(\omega - 2)$ , where  $\omega \geq 8$ , and only in  $V_\omega^3$ ; ... **(z)** for the members of  $k(2k).(\omega - 2)$ , where  $\omega \geq 2(k + 1)$ , and only in  $V_\omega^k$ , ( $k \geq 2$ ); **(4)**  $\lambda = 4$  happens in  $V_\omega$ : **(a)** for the members of  $246.(\omega - 3)$  where  $\omega \geq 9$ , and only in  $V_\omega^2$ ; **(b)** for the members of  $369.(\omega - 3)$  where  $\omega \geq 12$ , and only in  $V_\omega^3$ ; ... **(z)** for the members of  $k(2k)(3k).(\omega - 3)$ , where  $\omega \geq 3(k + 1)$ , and only in  $V_\omega^k$ , ( $k \geq 2$ ). The following result is obtained.

**Theorem 3** **(a)**  $\lambda = 1$  happens in  $V_\omega$ , and only for the strings of  $V_\omega$  starting at  $i_0$ ; **(b)** for each  $k \geq 2$ , any fixed  $\lambda > 1$  happens in  $V_\omega$  for the members of  $k(2k)(3k) \dots ((\lambda - 1)k).(\omega - \lambda + 1)$ , where  $\omega \geq (\lambda - 1)(k + 1)$ , and only for the subsets  $V_\omega^k$ .

Let  $W_\omega^k \subseteq V_\omega$  consist of the strings of length  $\lambda = k$  in the statement of Theorem 3. Then  $|W_\omega^1| = 1$  and  $|W_\omega^k| = 0$  whenever  $\omega < 3k$ , for  $k \geq 2$ . Moreover, if  $S_j^0 = 1$  and  $S_j^h = \sum_{k=1}^j S_k^{h-1}$ , ( $h > 0$ ), for every  $j \geq 1$ , so  $S_j^h - S_{j-1}^h + S_j^{h-1}$  for  $h > 0$  and  $j > 1$ , then

$$S_j^h = \binom{j+h-1}{h}, \quad |W_1^k| = S_1^k = k \quad \text{and in general} \quad (1)$$

$$|W_\omega^k| = \sum_{i=0}^{\lfloor \frac{\omega}{k+1} \rfloor} S_{\omega-i(k+1)}^k = \sum_{i=0}^{\lfloor \frac{\omega}{k+1} \rfloor} \binom{\omega - ik - i + k - 1}{k}, \quad (2)$$

for every weight  $\omega$  valid in  $ST_{\omega+1}$  and every string length  $k$ .

**Theorem 4** For  $0 < \omega \in \mathbf{Z}$ , the number of vertices of  $ST_{\omega+1}$  having weight  $\omega$  is given by the finite sum  $|V_\omega| = |W_\omega^1| + |W_\omega^2| + \dots + |W_\omega^k| + \dots$ .

It is easy to establish the following expression for the diameter  $D(n) = D(ST_n)$  of  $ST_n$ .

**Proposition 5** The diameter of  $ST_n$  is  $D(n) = \lfloor \frac{n-1}{2} \rfloor + n - 1$ .

Let  $V_\omega(n)$  be the set of vertices of  $\Lambda_n$  having weight  $\omega$ . Let  $W_\omega^k(n)$  be the subset of admissible strings corresponding to vertices of  $V_\omega(n)$  whose length  $\lambda$  is equal to  $k$ . Then, from the tables  $\mathcal{T}_n$  we get:

$$|W_\omega^k(n)| = |W_\omega^k|, \quad (0 \leq k < n); \quad (3)$$

$$|W_\omega^k(n)| = |W_\omega^k| - \sum_{j=0}^{k-n} |W_\omega^j|, \quad (n \leq k \leq D(n)). \quad (4)$$

The main result of the section follows.

**Theorem 6** *The cardinality of the set of vertices of  $ST_n$  having weight  $\omega$  is*

$$|V_\omega(n)| = |W_\omega^0(n)| + |W_\omega^1(n)| + \dots + |W_\omega^{D(n)}(n)| = \sum_{i=0}^{D(n)} W_\omega^i,$$

where the terms of the displayed sum are obtained by means of equations (1), (2), (3) and (4) presented above.

**Proof.** The equations and the statement of the theorem arise naturally from the patterns in the tables  $\mathcal{T}_n$  and the previous results.  $\square$

## 5 Weight distributions of E-sets in $ST_n$

It was proved in [3] that if  $1 \leq i \leq n$ , then, the vertex subset  $C_i$  of  $ST_n$  corresponding to the permutations  $\sigma_1 \sigma_2 \dots \sigma_n$  with a fixed  $\sigma_1 = i$  forms an E-set. This is the only way of getting an E-set in  $ST_n$ . Furthermore, it can be seen that the E-sets of  $ST_n$  form a partition of the vertex set of  $ST_n$ .

Having established in Section 4 the distribution of weights of vertices of  $ST_n$ , we ask, How does such a distribution restricts to each  $C_i$ ?

**Proposition 7** *The vertices  $u$  of  $\Lambda_n$  with  $\Sigma(u) = \sigma_1 \sigma_2 \dots \sigma_n$  and  $\sigma_1 = 1$  represent all the vertices of  $ST_n$  with  $\sigma_1 = 1$ . They have associated admissible strings  $i_0 i_1 \dots i_{j-1} i_j$  with  $i_{j-1} = i_j$ .*

**Proof.** This is clear from the developments above.  $\square$

Let  $V_\omega^i(n)$  be the set of vertices of  $C_i$  having weight  $\omega$  in  $ST_n$ , for  $1 \leq i \leq n$ .

**Theorem 8** *The weight distribution of the subsets  $C_i$  of  $ST_{n+1}$ , for  $2 \leq i \leq n+1$ , is given by:*

$$\begin{aligned} |V_0^i(n+1)| &= 0; \\ |V_\omega^i(n+1)| &= |V_{\omega-1}(n)|, \quad \text{for } \omega = 1, 2, \dots, 2 \lfloor \frac{D(n+1)}{2} \rfloor; \\ |V_{D(n+1)}^i(n+1)| &= 0, \quad \text{for } n \text{ even, (only case not covered above)} \end{aligned}$$

**Proof.** For each  $i \in \{2, \dots, n+1\}$ , the permutations  $\sigma_1 \sigma_2 \dots \sigma_{n+1}$  with  $\sigma_i = i$  induce a copy  $H_i$  of  $ST_n$  in  $ST_{n+1}$  containing the identity permutation  $12 \dots (n+1)$ . Each vertex  $h$  of  $H_i$  has a unique neighbor  $h^i$  in  $ST_{n+1} \setminus H_i$ . Then the collection of all  $h^i$  is  $C_i$ , for each  $i \in \{2, \dots, n+1\}$  fixed.  $\square$

**Remark.** According to Theorem 8, the  $n$  vertex subsets  $C_i$  in  $ST_{n+1}$  with  $1 < i \leq n+1$  have equivalent weight distributions. Thus, by multiplying the quantities obtained in the theorems by  $n$  and subtracting the results correspondingly from those obtained for  $ST_{n+1}$ , the case for  $C_1$  can be obtained, which uses that if  $n$  is odd then  $|V_{D(n)}| = (n-2)(n-4) \dots \times 5 \times 3$ , by Theorem 2.

## 6 Threading $\Lambda_n$ into an orientation of $ST_n$

We now modify the Pruning Algorithm into a threading algorithm in order to produce an orientation  $\Gamma_n$  of  $ST_n$  whose vertices are those of  $\Lambda_n$  (remaining after applying the algorithm) and whose arc set contains the arc set of  $\Lambda_n$ .

The Threading Algorithm consists in running the Pruning Algorithm (on the previously defined  $\Lambda_n$ ), checking whether the last field  $b_u a_u$  of each line in the table  $\mathcal{P}_n$  that is being generated has  $b_u = 0$  and  $a_u \geq 2$ . If this is the case, then a *thread*, meaning a new arc, is added to  $\Lambda_n$  from  $u$  to a vertex  $\psi(u)$  determined as follows. It happens that the penultimate field  $C(u)$  was present in a previous line of  $\mathcal{P}_n$  corresponding to the tail  $\phi(u)$  of a vertical arc  $e(u)$  of  $\Lambda_n$  having head  $\psi(u)$ . Then  $\psi(u)$  is the head of  $e(u)$ .

**Example.** Working with  $\mathcal{P}_9$ , the threads appearing by means of the Threading Algorithm are departing from the vertices  $u$  with subindex strings 35, 46, 47, 57, 58, 68, 257, 268, 368, whose values  $C(u)$  are respectively 32, 42, 43, 52, 53, 62, 232, 242, 332 and whose fields  $b_u a_u = 0 a_u$  have  $a_u = 2, 2, 3, 2, 3, 2, 2, 2, 2$ , respectively. But the vertices  $\phi(u)$  with respective subindex strings 25, 26, 37, 27, 38, 28, 257, 268, 368, have the same corresponding values  $C(u)$ , presented in  $\mathcal{P}_9$  in nondecreasing order: 23, 24, 34, 25, 35, 26, 223, 224, 233, so the corresponding 1-ics's are the same in both cases. We obtain the desired orientation of  $ST_9$  by adding a thread from each one of the eight mentioned vertices respectively into the vertices  $\psi(u)$  whose subindex strings are 255, 266, 377, 277, 388, 288, 2477, 2488, 2588, which are the heads of the respective arcs  $e(u)$  (that departed from the vertices  $\phi(u)$  mentioned above).

**Theorem 9** *Any pair  $(u, \phi(u))$  appearing during the running of the Threading Algorithm has the vertices  $u$  and  $\phi(u)$  with  $C(u) = C(\phi(u))$ , where the order of the elements on each side of the equality is irrelevant. Thus, in the running of the Threading Algorithm, each consideration of a vertex  $u$  of  $\Lambda_n$  with  $C(u)$  equal to the  $C(v)$  of a previously considered vertex  $v = \phi(u)$  determines a thread from  $u$  onto the corresponding  $\psi(u)$ .*

**Proof.** The statement follows from the previous discussion and Theorem 1, item 1.  $\square$

**Remark.** The Threading Algorithm insured by Theorem 9 produces an orientation  $\Gamma_n$  of  $ST_n$  whose vertices represent the 1-ics's of the permutations on  $n$  elements, that is each vertex of  $\Gamma_n$  represents all the permutations on  $n$  elements having a specific 1-ics, and there is a bijective correspondence between the vertices of  $\Lambda_n$  and the 1-ics's of permutations on  $n$  elements. Thus  $\Gamma_n$  may be referred to as the 1-ics orientation of  $ST_n$ . Each arc of  $ST_n$  projects into a specific arc of  $\Gamma_n$ . We still consider that the arcs of  $\Gamma_n$  are 'horizontal' and 'vertical', as in the case of  $\Lambda_n$ , where threads of  $\Gamma_n$  are 'vertical'. Moreover, the vertices and arcs of  $\Gamma_n$  may be considered as preserving the indications they inherit from  $\Lambda_n$ , including the threads, which preserve the indications of the arcs removed by the Pruning Algorithm. As said above, the indications of horizontal arcs are of the form  $\bullet\ell_u$ , so we still have that the orientations  $\Gamma_n$  form a nested sequence of indicated digraphs and that their limit indicated digraph  $\Gamma_\infty$  is well defined and constitutes a universal graph for this situation. This corresponds to the infinite star graph  $ST_\infty$  that can be defined as the Cayley graph of the symmetric group  $S_\infty$  with respect to the set of transpositions  $\Theta_\infty = \{(1\ i),\ i = 2, \dots, n, \dots\}$ .

**Theorem 10**  $\Gamma_n$  can be interpreted as an orientation of  $ST_n$  via the map  $\Phi_n : ST_n \rightarrow \Lambda_n$  given by  $\Phi_n^{-1}(u) = \rho$ -equivalence class of  $\Sigma(u)$ , for each vertex  $u$  of  $\Lambda_n$ . Then: **(1)** the value  $c(u)$  of each vertex  $u$  of  $\Gamma_n$  is the cardinality of  $\Phi_n^{-1}(u)$  and **(2)** the inverse image  $\Phi_n^{-1}$  of an horizontal, (vertical), arc  $e$  of  $\Lambda_n$  is formed by  $c(u^e)$ ,  $(c(u_e))$ , arcs subdivided into  $c(u^e)/m_e$ ,  $(c(u_e)/d_e)$ , subsets of  $m_e$ ,  $(d_e)$ , arcs incident each to a common corresponding vertex in  $\Phi_n^{-1}(u_e)$ ,  $(\Phi_n^{-1}(u^e))$ .

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